AdS bubbles, entropy and closed string tachyons

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ABSTRACT: We study the conjectured connection between AdS bubbles (AdS solitons) and closed string tachyon condensations. We confirm that the entanglement entropy, which measures the degree of freedom, decreases under the tachyon condensation. The entropies in supergravity and free Yang-Mills agree with each other remarkably. Next we consider the tachyon condensation on the AdS twisted circle and argue that its endpoint is given by the twisted AdS bubble, defined by the double Wick rotation of rotating black 3-brane solutions. We calculated the Casimir energy and entropy and checked the agreements between the gauge and gravity results. Finally we show an infinite boost of a null linear dilaton theory with a tachyon wall (or bubble), leads to a solvable time-dependent background with a bulk tachyon condensation. This is the simplest example of spacetimes with null boundaries in string theory.

Contents

1. Introduction

The understandings of closed string tachyon condensation have been far from complete in spite of many efforts e.g. the pioneering work by Adams, Polchinski and Silverstein [1] (a list of references can be found in reviews [2, 3]). One of the most important problems is to examine the time-dependent dynamical process of tachyon condensation.

Recently, a remarkable progress has been made by Horowitz and Silverstein [4, 5] employing the AdS/CFT correspondence [6]. They considered an unstable configuration of the near horizon geometry of D3-branes by putting the anti-periodic boundary condition for fermions. There appears a closed string tachyon which is localized in a finite region

of the spacetime [7, 4, 8, 5]. The endpoint of the tachyon condensation is conjectured to be the static AdS bubble solution (or AdS soliton) [4, 5]. This process of tachyon condensation has an equivalent description in the dual Yang-Mills theory. The dynamics of tachyon condensation is mapped to the more traditional problem of time-dependent process in strongly coupled gauge theories.

In this paper, we would like to explore this scenario further. First we show that the degree of freedom decreases under this tachyon condensation process by computing the entanglement entropy [9–11] of the dual Yang-Mills theory.[1] In general, the degree of freedom is expected to decrease under the closed string tachyon condensation since the radiations produced by the process will carry away a part of it. We will also present the dual holographic computation[2] [11] in supergravity. This analysis of the entanglement entropy offers a new evidence for the conjectured scenario of the closed string tachyon condensation, in addition to the known decreasing of energy density [13, 5].

Next we consider a near horizon geometry of D3-branes with a twisted boundary condition. The dual geometry is described by the AdS geometry with the twisted identification. It is equivalent to a twisted circle or Melvin background (refer to e.g. the papers [14–23, 3]) fibred over the radial direction of the AdS. Its radius of the circle shrinks toward the IR region. Then the closed string theory in this background has a tachyon field localized both in the IR region and in a certain S^3 inside the S^5. We will claim that the end point of the tachyon condensation is given by the bubble solution obtained from the double Wick rotation of the rotating D3-brane solution. We will check this claim by computing the energy density (Casimir energy) and entanglement entropy in both the free Yang-Mills and gravity theory. We find qualitative agreements between them in general. Remarkably, the entropy in the near extremal region precisely agrees with each other including the numerical factor. We may also think this as a further evidence for AdS/CFT correspondence in a slightly non-BPS background. We also observe the qualitative agreement between the ADM energy of the twisted AdS bubble and the Casimir energy of the dual free Yang-Mills.

Another way to study the dynamical process of tachyon condensation is to directly construct the corresponding time-dependent backgrounds of string theory. Recently, its possible relevance to a resolution of cosmological singularities has been discussed in [24]. In general, it is very difficult to find a well-controllable time-dependent model with closed string tachyon condensation.

A simple example of the bulk tachyon condensation in bosonic string (or type 0 string) has been proposed to be described by the time-like Liouville theory [25, 26] (see e.g. [27–32, 24, 5] for a partial list of further progresses). However, this theory has not been completely understood, especially because the continuation from the Euclidean theory is not straightforward; there is a potential ambiguity with respect to the choice of vacua in a time-dependent background. In the last part of the present paper, we give a simpler solvable

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[1]Notice that the thermodynamical entropy is zero for this solution since we are considering zero temperature.

[2]Quite recently, a slightly analogous holographic relation about the entanglement entropy in 3D topological QFTs has been pointed out in [12]. There, the boundary entropy (or $g-$function) is dual to the bulk topological entanglement entropy.

model which describes bulk closed string tachyon condensation. This is obtained by the infinite boost of the null linear dilaton background with a Liouville potential. Equivalently we can regard the background as a flat spacetime with a null boundary. Since there has been no general answer to what kinds of boundaries are allowed in a spacetime of string theory, this offers an useful basic example.

After we completed computations in this paper, we noticed an interesting paper [33] which discusses the analogous scenario of the closed string tachyon condensation via the AdS/CFT correspondence in a different model [34] (see also [35]).

The organization of this paper is as follows. In section 2, we first review the conjectured connection between AdS bubbles and closed string tachyon condensation. Then we provide a further evidence for this conjecture by computing the entanglement entropy. In section 3, we consider the twisted AdS bubble solution and claim that it is the endpoint of the closed string tachyon condensation on D3-branes wrapped on the twisted circle by computing the energy and entropy. In section 4, we present a simple construction of a spacetime with a null boundary via bulk closed string tachyon condensation. In section 5 we summarize conclusions.

2. AdS bubbles and closed string tachyons

2.1 Static AdS bubble solution

Consider N D3-branes in type IIB string. The world volume coordinates are denoted by (t, χ, x_1, x_2). We compactify χ with period L and put the anti-periodic boundary condition for all fermions. Its near horizon geometry is represented by the AdS_5

$$ds^2 = R^2 \frac{dr^2}{r^2} + \frac{r^2}{R^2}(-dt^2 + d\chi^2 + dx_1^2 + dx_2^2). \tag{2.1}$$

The important point is that the radius of the thermal circle χ gets smaller as we goes into the IR region $r \to 0$. Thus we expect that when its radius $\frac{rL}{R}$ is of order l_s (i.e. string scale), a closed string tachyon appears. This tachyon is clearly localized in the IR region. To make this more precise, we can start with a shell of D3-branes which is described by the ten dimensional metric of type IIB supergravity.

$$ds^2 = h^{-1}(r)[-dt^2 + d\chi^2 + dx_1^2 + dx_2^2] + h(r)(dr^2 + r^2 d\Omega_5^2), \tag{2.2}$$

where $h(r)$ is the function defined by

$$h(r) = \frac{R^2}{r^2} \ \ (r > r_0), \quad h(r) = \frac{R^2}{r_0^2} \ \ (r \le r_0). \tag{2.3}$$

Then the inside of the shell ($r < r_0$) the metric is flat and thus we can employ the familiar perturbative world-sheet analysis on the existence of closed string tachyons.

The remarkable claim made by Horowitz and Silverstein [5] is that this unstable background decays into the static bubble [36]

$$ds^2 = R^2 \frac{dr^2}{r^2 f(r)} + \frac{r^2}{R^2}(-dt^2 + f(r)d\chi^2 + dx_1^2 + dx_2^2), \tag{2.4}$$

where $f(r) = 1 - (r_0/r)^4$. This can be obtained from the double wick rotation of the AdS-Schwartzschild solution and is called the (static) AdS bubble[3] or AdS soliton [13]. Near the point $r = r_0$ the metric is approximated by $ds^2 \simeq dy^2 + \frac{4r_0^2}{R^4} y^2 d\chi^2$, where $y^2 \equiv R^2(r-r_0)/r_0$. Thus to make the metric regular at this point the periodicity L of χ should be

$$L = \frac{\pi R^2}{r_0}. \tag{2.5}$$

In the dual Yang-Mills theory, this closed string tachyon condensation is interpreted as follows [5]. The near horizon limit of the D3-brane shell (2.2) corresponds to the supersymmetric vacuum of $N = 4$ super Yang-Mills theory with non-zero expectation values of transverse scalar fields. Now we compactify one of the three space coordinates and put the anti-periodic boundary condition for all fermions. Then the supersymmetry is completely broken and the scalar fields acquire non-zero masses from radiative corrections. Thus the Coulomb branch is lifted and the theory becomes almost the same as the pure Yang-Mills, which shows the confinement behavior [36]. The cut off of the IR region $r > r_0$ in the bubble solution (2.4) corresponds to the mass gap due to this confinement.

2.2 Casimir energy

An important evidence for this conjecture is that the AdS soliton has the lowest energy[4] [13, 5] given by

$$\frac{E}{V_2} = -\frac{\pi^3 R^3}{16 G_N^{(5)} L^3} = -\frac{\pi^2}{8} \cdot \frac{N^2}{L^3}, \tag{2.6}$$

where[5] $G_N^{(5)}$ is the 5D Newton constant, and V_2 is the infinite volume of (x_1, x_2). Here we have used the definition of the energy in an asymptotically AdS space [40, 41]

$$E = -\frac{1}{8\pi G_N^{(5)}} \int_S N(K - K_0), \tag{2.7}$$

where the integral is over a surface near infinity S. N is defined such that the norm of the time-like Killing field is $-N^2$. K is the trace of the extrinsic curvature of this surface. K_0 is the trace of the extrinsic curvature of a surface with the same intrinsic geometry in the background spacetime. The energy in an asymptotically AdS space is defined such that the AdS space itself has the vanishing energy $E = 0$ [40, 13].

It is useful to compare the above energy with the Casimir energy computed in the free Yang-Mills theory [13, 42]. Consider a massless real scalar field[6] $\phi(t, \chi, x_1, x_2)(= \phi(x))$. We compactify the χ direction such that $\chi \sim \chi + L$. Then the two point function can be found to be

$$\langle \phi(x)\phi(x') \rangle = \frac{1}{4\pi^2} \sum_{n \in \mathbf{Z}} \frac{1}{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (\chi - \chi' - nL)^2 - (t - t')^2}. \tag{2.8}$$

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[3]Refer to e.g. [37, 38] for a time-dependent bubble solution obtained by another double Wick rotation of the AdS black hole.

[4]Refer to [39] for the proof that the energy is decreasing under the closed string tachyon condensation in asymptotically flat spaces.

[5]Also we have employed the standard relation $G^{(5)} = \frac{\pi R^3}{2N^2} = \frac{G^{(10)}}{\pi^3 R^5}$ in the type IIB string on $AdS_5 \times S^5$.

[6]We normalized the field such that the Lagrangian is given by $L = \frac{1}{2}(\partial_\mu \phi)^2$.

The energy density $T_{00} = \frac{1}{2}\left[(\partial_0 \phi)^2 + \sum_i (\partial_i \phi)^2\right]$ can be obtained by the point splitting regularization (refer to [43, 42] for details)

$$\lim_{x \to x'} \frac{1}{2}\left[\partial_0 \phi(x)\partial_0' \phi(x') + \partial_i \phi(x)\partial_i' \phi(x')\right]. \tag{2.9}$$

This leads to

$$T_{00} = \frac{1}{4\pi^2} \sum_{n \neq 0} \left(-\frac{2}{(Ln)^4}\right) = -\frac{\pi^2}{90L^4}, \tag{2.10}$$

where we regularize the summation by excluding the divergent term $n = 0$. We can perform the similar analysis for a free Majorana fermion ψ with the anti-periodic boundary condition $\psi(z + L) = -\psi(z)$ and obtain the energy density for each component

$$T_{00} = \frac{1}{4\pi^2} \sum_{n \neq 0} \left(\frac{2(-1)^n}{(nL)^4}\right) = -\frac{7\pi^2}{720L^4}. \tag{2.11}$$

In the free N=4 super Yang-Mills, there are $8N^2$ bosons and $8N^2$ fermions and thus we finally obtain[7]

$$\frac{E}{V_2 L} = T_{00} = 8N^2 \cdot \left(-\frac{\pi^2}{90L^4}\right) + 8N^2 \cdot \left(-\frac{7\pi^2}{720L^4}\right) = -\frac{\pi^2 N^2}{6L^4}. \tag{2.12}$$

This agrees with (2.6) up to the factor 4/3 [13]. Since the gravity description corresponds to the strongly coupled limit of the Yang-Mills theory, we can say that this agreement is rather excellent.

2.3 Entanglement entropy: gravity side

Now we wish to turn to another quantity called the entanglement entropy [9 – 11] as another evidence for the closed string tachyon condensation. Divide the space manifold (in our case it is $R^2 \times S^1$) into two parts A and B, and trace out the Hilbert space for the subsystem B. This procedure defines the reduced density matrix ρ_A for the subsystem A. Then the entanglement entropy S_A is defined by the von-Neumann entropy $S_A = -\text{tr}\rho_A \log \rho_A$ with respect to the reduced density matrix ρ_A. This leads to a non-vanishing entropy even if we start with a pure state on the total space $A \cup B$. The choice of A is arbitrary and we can define infinitely many entropies S_A accordingly.

In general, the entanglement entropy measures the degree of freedom and thus we would like to claim that *the entanglement entropy in the dual Yang-Mills theory should decrease under the closed string tachyon condensation*. Indeed, in two dimension the entropy is essentially known to be proportional to the central charge c [44, 45]. However, we should keep in mind that the UV behavior of S_A will not change under the localized closed string tachyon condensation. Thus the divergent piece of the entropy, which is proportional to the area of the boundary ∂A of the subsystem A (known as the area law [9, 10]), will not change because this part is only sensitive to UV quantities. We expect that only a finite

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[7]Here we are implicitly using the fact that the contribution of the gauge fields is the same as that of two real scalar fields.

part of the entropy will change. Thus we will consider the difference between the entropy before and after the tachyon condensation.

In our setup of asymptotically AdS spaces we can apply the holographic computation of entanglement entropy [11]. The entropy is given by the formula [11]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(5)}}, \tag{2.13}$$

where $\text{Area}(\gamma_A)$ is the area of the minimal surface γ_A whose boundary coincides with ∂A. Refer to [46] for its proof from the basic principle of the AdS/CFT correspondence. Its interpretation from the viewpoint of the entropy bound (Bousso bound) is given in [47].

First we assume the subsystem A is defined by $x_1 > 0$ and extends in the x_2 and χ direction. Then the entropy can be found from (2.13) (we put the UV cutoff[8] $r < r_\infty$)

$$S_A = \frac{V_1 L}{4G_N^{(5)}} \int_{r_0}^{r_\infty} dr \frac{r}{R} = \frac{\pi R V_1 r_\infty^2}{8G_N^{(5)} r_0} - \frac{\pi R V_1 r_0}{8G_N^{(5)}}. \tag{2.14}$$

The first term is divergent and represents the area law [9, 10]. It is the same as the one in the original AdS_5 background. The second term in (2.14)

$$\Delta S_A = -\frac{\pi R V_1 r_0}{8G_N^{(5)}} = -\frac{\pi N^2 V_1}{4L}, \tag{2.15}$$

does not depend on the cut off and thus is physically important. This is equal to the difference between the entropy in the AdS bubble and the one in the pure AdS. Since it is negative, we find that the entropy of the AdS bubble is decreased compared with the AdS_5 solution as we expected. Furthermore, we would like to conjecture that the AdS bubble has the lowest value among other asymptotically AdS solution with the same symmetry. This is because from the energy analysis it is considered to be the lowest energy configuration [13] and thus is the most stable solution.

For example, we can consider the time-dependent bubble solution[9] found in [5] , which has a larger energy. The metric of the constant time slice of this solution at $t = 0$ is given by

$$ds_{t=0}^2 = \left(\frac{r^2}{R^2} - \frac{r_0^4}{R^2 r^2} \right) d\chi^2 + \frac{dr^2}{\left(\frac{r^2}{R^2} - \frac{r_0^4}{R^2 r^2} \right) \left(1 + \frac{b}{3r^4 - r_0^4} \right)} + \frac{r^2}{R^2} (dx_1^2 + dx_2^2). \tag{2.16}$$

The specific point $b = 0$ is the same as the AdS bubble with the lowest energy. We have numerically checked that the entropy S_A at $t = 0$ computed in the same way as in (2.14) for $b > 0$ always takes a larger value than that of the AdS bubble $b = 0$.

It is also useful to examine the entropy in the shell configuration (2.2) since to make sure the existence of tachyon it is better to start with the shell background of D3-branes. In

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[8]Relation to the lattice spacing introduced in [11] is given by $r_\infty = R^2/a$.

[9]We are very grateful to Gary Horowitz for pointing out an important error in this paragraph of the first version of our paper.

this case the entropy can be found from the integration over the codimension three surface γ_A, which is similar to the previous one, times S^5 as follows [11]

$$
\begin{aligned}
S_A &= \frac{1}{4G_N^{(10)}} \int_{\gamma_A \times S^5} dx^8 \sqrt{g} \\
&= \frac{V_1 L}{4G_N^{(10)}} \int_0^{r_\infty} dr\, r^5 h(r)^2 \\
&= \frac{V_1 L}{4G_N^{(5)} R} \left[\frac{r_\infty^2}{2} - \frac{r_0^2}{3} \right],
\end{aligned} \tag{2.17}
$$

where $G_N^{(10)}$ is the 10D Newton constant. This is clearly larger than the entropy in the AdS bubble (2.14) and thus the difference of the entropies again becomes negative

$$
\Delta S_{A(shell)} = -\frac{\pi R V_1 r_0}{24 G_N^{(5)}} = -\frac{\pi N^2 V_1}{12 L} < 0. \tag{2.18}
$$

This supports the conjecture that the AdS shell decays into the AdS bubble.

Now it is also possible to compute S_A when the subsystem A is a straight belt with a finite width l. Suppose A is defined by $-l/2 \leq x_1 \leq l/2$, $0 \leq x_2 \leq V_1(\to \infty)$ and $0 \leq \chi \leq L$. In the dual AdS gravity, we need to consider a minimal surface γ_A whose boundary (i.e. $r \to \infty$) coincides with the boundary ∂A of A.

The area can be written as

$$
\text{Area} = L V_1 \int_{-l/2}^{l/2} dx_1 \frac{r}{R} \sqrt{\left(\frac{dr}{dx_1} \right)^2 + \frac{r^4 f(r)}{R^4}}. \tag{2.19}
$$

The energy conservation leads to

$$
\frac{dr}{dx_1} = \frac{r^2}{R^2} \sqrt{f(r) \left(\frac{r^6 f(r)}{r_*^6 f(r_*)} - 1 \right)}, \tag{2.20}
$$

where r_* is the minimal value of r. The relation between r_* and l is fixed by

$$
\frac{l}{2} = \int_{r_*}^\infty dr \frac{R^2}{r^2 \sqrt{f(r) \left(\frac{r^6 f(r)}{r_*^6 f(r_*)} - 1 \right)}}. \tag{2.21}
$$

Finally the entropy can be found as

$$
S_A = \frac{L V_1}{2 R G_N^{(5)}} \int_{r_*}^\infty \frac{r^4 \sqrt{f(r)}}{\sqrt{r^6 f(r) - r_*^6 f(r_*)}}. \tag{2.22}
$$

An important point is that when we change the values of r_* arbitrary, only the following specific values of l is allowed by the relation (2.21)

$$
l \leq l_0 \simeq 0.69 \cdot \frac{R^2}{r_0} \simeq 0.22 \cdot L. \tag{2.23}
$$

Thus when l is large enough, there is no minimal surface that connects the two boundaries of ∂A. Under this situation $l > l_0$, the minimal surface is given by the disconnected sum of the ones considered in (2.14).

The explicit form of the entropy as a function of l is presented in figure 1. We subtracted the entropy S_A^{trivial} for the trivial disconnected solution (=twice of (2.14)). The lower solution in figure 1 is physical compared with the upper one because it has the lower entropy and gives a dominant contribution to the path-integral in the gravity. When $S_A - S_A^{\text{trivial}}$ becomes positive ($\tilde{l}_1 \sim 0.31$), the physical solution is replaced by the trivial disconnected one. Thus there is a phase transition[10] at a specific value $l_1 (< l_0)$. This is not surprising since there will be no correlation between a distance larger than $\sim L$ due to the mass gap $\sim 1/L$.
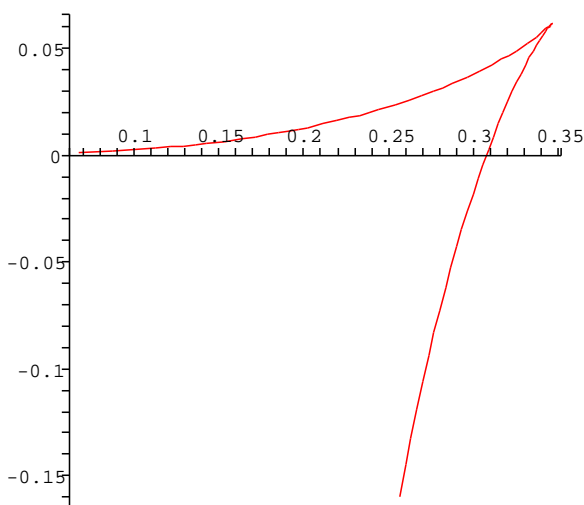

Figure 1: The entanglement entropy as a function of the width l of the subsystem A. We subtracted the entropy S_A^{trivial} for the trivial disconnected solution (=twice of (2.14)). We plotted the function $s = s(\tilde{l})$ defined by $S_A - S_A^{\text{trivial}} = \frac{LV_1 r_0^2}{2RG_N^{(5)}} s(l)$ and $\tilde{l} = \frac{\pi}{2L} l$. The lower solution is physical compared with the upper one because it has the lower entropy.

We would also like to notice that the physical solution in figure 1 (i.e. lower one) is concave as a function of l or equally $\frac{d^2 S_A}{dl^2} \leq 0$. This follows from the general property of the von-Neumann entropy, which is known as the strong subadditivity [47].

In general 4D conformal field theories, the entanglement entropy defined in the same way takes the following form [48, 11]

$$S_A = \gamma \cdot \frac{LV_1}{a^2} - C \cdot \frac{LV_1}{2l^2}, \quad (2.24)$$

where γ and C are numerical constants which are proportional to the number of fields. The first term in (2.24) represents the area law divergent term [9 – 11]. The second finite term is more interesting because it does not depend on the UV cutoff $a \to 0$. Motivated by this, we would like to call the following quantity an entropic c-function

$$C(l) = \frac{l^3}{LV_1} \cdot \frac{dS_A(l)}{dl}. \quad (2.25)$$

This is a natural generalization of the entropic c-function defined in two dimension [49, 48]. Its explicit form is plotted in figure 2. Since l corresponds to the length scale which we are looking at, $C(l)$ measures the degree of freedom at the energy scale $\sim l^{-1}$ in the given field theory. The figure 2 shows that its value monotonically decreases as we decrease the energy scale from the UV region. It jumps to zero in the middle point and it continues to be

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[10]A quite similar discontinuity has been found in the holographic computation of entanglement entropy defined by the annular boundary in $N = 4$ super Yang-Mills [47].

vanishing in the IR region. These are consistent with the expected property of c-function that it decreases[11] under the RG-flow. The sharp dump of $C(l)$ is because the IR region $r < r_0$ is completely cutoff in the AdS bubble solution and represents the clear mass gap in the dual CFT.

2.4 Entanglement entropy: free Yang-Mills analysis

Next we would like to compare the above results from the gravity side with the direct Yang-Mills free field computations.
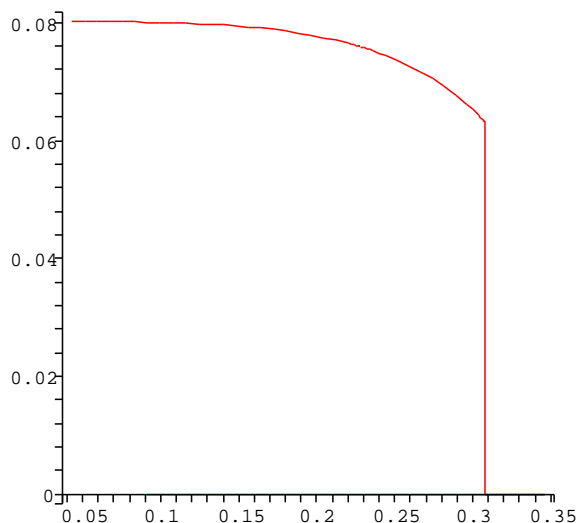

Figure 2: The form of the entropic c-function. We plotted $\tilde{l}^3 \frac{ds(\tilde{l})}{dl}$ as a function of $\tilde{l}$. It jumps to zero at $\tilde{l} = \tilde{l}_1 \sim 0.31$ and for larger values of $\tilde{l}$ it is given by zero.

Consider a free massless scalar field in $R^{1,d-1} \times S^1$. The radius of the circle S^1 is L. We divide the space manifold $R^{d-1} \times S^1$ (the coordinates are denoted by $(x^1, x^2, \cdots, x^{d-1}; x^d)$) into the submanifolds A and B such that the boundary $\partial A = \partial B$ is given by $R^{d-2} \times S^1$ defined by $x_1 = 0$.

The entanglement entropy can be evaluated as follows (see e.g. [50, 51, 45, 11, 46]). First we compute the partition function $Z_n = e^{-\beta F}$ on the Euclidean manifold $M_n = \Sigma_n \times R^{d-2} \times S^1$. The 2D manifold Σ_n is the n-sheeted Riemann surface defined by the metric $ds^2 = d\rho^2 + \rho^2 d\theta^2$ in the polar coordinate and the conical periodicity $0 \leq \theta \leq \beta = 2\pi n$. Then the entropy is obtained by

$$S_A = \left[\beta \frac{\partial}{\partial \beta} - 1 \right] (\beta F)|_{\beta=2\pi}, \quad (2.26)$$

as if β were a real temperature.

One conventional way to compute the free energy F is to employ the heat kernel method [51]. Instead, here, we will calculate F using an orbifold theoretic analysis. In the computation of the entropy, we can equally consider the positive deficit angle instead of the negative one. One such example is the orbifold C/Z_N. In the setup of QFT on $R^{1,d}$, the partition function of a free massive scalar on $C/Z_N \times R^{d-1}$ can be found as

$$\log Z_{C/Z_N} = V_{d-1} \int_0^\infty \frac{ds}{2s} \cdot \left[\int \left(\frac{dk_\perp}{2\pi} \right)^{d-1} e^{-sk_\perp^2} \right] \cdot \frac{1}{N} \sum_{k=0}^{N-1} \text{Tr} \left[g^k \cdot e^{-m^2 s} \right]$$

$$= V_{d-1} \int_0^\infty \frac{ds}{2s} \frac{1}{(4\pi s)^{\frac{d-1}{2}}} \cdot \frac{1}{N} \sum_{k=0}^{N-1} \text{Tr} \left[g^k \cdot e^{-m^2 s} \right]. \quad (2.27)$$

This formula is easily obtained by remembering the expression of the open string cylinder amplitude (see e.g. the Polchinski's text book [52]) in the Schwinger representation. Tr

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[11]This is very similar to the celebrated Zamolodchikov's c-theorem. Also notice that this is the c-theorem for the spacetime field theories on D-branes. Do not confuse this with the c-theorem for the RG-flow of the string world-sheet theory, which often appears when we discuss closed string tachyon condensations.

in (2.27) denotes the trace over the zeromodes (the coordinate $(z, \bar{z})$ and their momenta) of the fields on $R^2 (= C)$. $k_\perp$ are the momenta in the transverse directions R^{d-1}.

The summation over k can be done exactly as follows

$$
\begin{aligned}
\frac{1}{N} \sum_{k=0}^{N-1} \text{Tr} g^k &= \frac{1}{N} \sum_{k=0}^{N-1} \int dz d\bar{z} \cdot \delta(z - e^{2\pi i k/N} z) \delta(\bar{z} - e^{-2\pi i k/N} \bar{z}) \\
&= \frac{1}{N} \int_{R^2} dz^2 \int \frac{dk^2}{(2\pi)^2} + \sum_{k=1}^{N-1} \frac{1}{4N \sin^2(\pi k/N)} \\
&= \text{Vol}(C/Z_N) \cdot \int \frac{dk^2}{(2\pi)^2} + \frac{1}{12}(N - 1/N).
\end{aligned} \tag{2.28}
$$

Thus we can reproduce the following known expression of the entropy (notice $n = 1/N$)

$$
\begin{aligned}
S_A &= -\frac{\partial}{\partial(1/N)} \left[\log Z_{C/Z_N} - \frac{\log Z_C}{N} \right]_{N=1} \\
&= \frac{\partial}{\partial N} \left[\frac{1}{12}(N - 1/N) \cdot V_{d-1} \int \frac{ds}{2s} \frac{1}{(4\pi s)^{\frac{d-1}{2}}} e^{-m^2 s} \right]_{N=1} \\
&= \frac{\pi}{3} V_{d-1} \int \frac{ds}{(4\pi s)^{\frac{d+1}{2}}} e^{-m^2 s}.
\end{aligned} \tag{2.29}
$$

In our setup, we would like to assume one of the transverse directions is compactified at the radius $\frac{L}{2\pi}$. Then we obtain

$$
S_A = \frac{\pi}{3} V_{d-2} \int_{a^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}} \cdot \frac{L}{2\pi} \cdot \sqrt{\frac{\pi}{s}} \cdot \sum_{q=-\infty}^{\infty} e^{-\frac{L^2 q^2}{4s}}, \tag{2.30}
$$

where we introduced the UV cutoff (or the lattice spacing) a. We divide (2.30) into the divergent $q = 0$ term and the finite $q \neq 0$ term

$$
S_A = S_A^{\text{area law}} + S_A^{\text{finite}}, \tag{2.31}
$$

$$
S_A^{\text{area law}} = \frac{\pi}{3} \cdot L V_{d-2} \cdot \int_{a^2}^{\infty} \frac{ds}{(4\pi s)^{(d+1)/2}}, \tag{2.32}
$$

$$
S_A^{\text{finite}} = \frac{V_{d-2}}{3} \cdot 2^{-1} \pi^{\frac{1}{2} - \frac{1}{2}d} \cdot \Gamma\left(\frac{d}{2} - \frac{1}{2}\right) \cdot \zeta(d-1) \cdot \frac{1}{L^{d-2}}. \tag{2.33}
$$

In particular, the 4D massless scalar $d = 3$, we find

$$
S_A^{\text{finite}} = \frac{\pi V_1}{36 L}. \tag{2.34}
$$

Now we turn to free fermions. By direct computation we can show that the expression of S_A is the same form and its coefficient is proportional to the central charge c when we reduce the system to two dimension [51]. One way to understand this is to note that the entropy of the $d + 1$ dimensional free field theory is obtained from the two dimensional entropy with the correlation length $\xi = 1/m$ [45]

$$
S_A = \frac{\pi c}{3} \int_{a^2}^{\infty} \frac{ds}{4\pi s} e^{-sm^2} = \frac{c}{6} \log(ma), \tag{2.35}
$$

by summing over the KK modes as follows

$$S_A = \frac{\pi c}{3} V_{d-2} \int \left(\frac{dk_\perp}{2\pi}\right)^{d-1} \int_{a^2}^\infty \frac{ds}{4\pi s} e^{-s(m^2 + k_\perp^2)} = \frac{\pi c}{3} V_{d-2} \int_{a^2}^\infty \frac{ds}{(4\pi s)^{(d+1)/2}} e^{-sm^2}. \quad (2.36)$$

Thus for each real component of a Majorana fermion with the periodic boundary condition we find

$$S_A^{(P)} = \frac{\pi}{6} V_{d-2} \int_{a^2}^\infty \frac{ds}{(4\pi s)^{d/2}} \cdot \frac{L}{2\pi} \cdot \sqrt{\frac{\pi}{s}} \cdot \sum_{q=-\infty}^\infty e^{-\frac{L^2 q^2}{4s}}. \quad (2.37)$$

On the other hand, the anti-periodic fermion leads to

$$S_A^{(A)} = \frac{\pi}{6} V_{d-2} \int_{a^2}^\infty \frac{ds}{(4\pi s)^{d/2}} \cdot \frac{L}{2\pi} \cdot \sqrt{\frac{\pi}{s}} \cdot \sum_{q=-\infty}^\infty (-1)^q \cdot e^{-\frac{L^2 q^2}{4s}}. \quad (2.38)$$

When $d = 3$, the finite part of the entropy for each of them is given by

$$S_A^{finite(P)} = \frac{\pi V_1}{72L}, \qquad S_A^{finite(A)} = -\frac{\pi V_1}{144L}. \quad (2.39)$$

In summary, the finite part of the entanglement entropy in N=4 SU(N) super Yang-Mills theory can be found in both periodic and anti-periodic fermion cases as follow

$$S_A^{finite(P)} = \frac{N^2 \pi V_1}{3L}, \qquad S_A^{finite(A)} = \frac{N^2 \pi V_1}{6L}. \quad (2.40)$$

Their difference

$$\Delta S_A = S_A^{(A)} - S_A^{(P)} = -\frac{\pi N^2 V_1}{6L}, \quad (2.41)$$

should be compared with the AdS result (2.15). They differs only by the factor $\frac{2}{3}$. Again, we can think this a successful agreement since the gravity calculation is dual to the strongly coupled Yang-Mills, while our gauge theoretic result is found for the free Yang-Mills. The entanglement entropy is not protected by any supersymmetries because the conical geometry appears in the definition (2.26) breaks all of supersymmetries.

Finally it may be interesting to examine the entanglement entropy in the free Yang-Mills when the subsystem A is defined by the straight belt with a finite width as in (2.24). This has been done in [11] for the N=4 super Yang-Mills and the entropy of the form (2.24) was obtained. The strategy is to first regard the system to infinitely many 2D free field theories and then to integrate the known numerical results of the 2D entropic c-function [48]. We can repeat the same computation in our compactified Yang-Mills theory. However, it does not seem to be possible to reproduce the phase transition as a function l found in the previous gravity analysis. This will be essentially because the phase transition occurs due to the strongly coupled phenomena (i.e. confinement), while we are treating the free Yang-Mills.

3. Twisted AdS bubbles and closed string tahcyons

We would like to study the second example i.e. the double Wick rotation of the rotating non-extremal D3-branes [53 – 55]. We will claim that this solution (we call it the twisted AdS bubble[12]) is the end point of the decay of the D3-brane background with a twisted boundary condition (i.e. the twisted circle or Melvin background [14, 22]). By taking the near horizon limit, this is equivalent to the statement that the AdS with the twisted identification (we call it the AdS twisted circle) will decay into the twisted AdS bubble via closed string tachyon condensation.

3.1 Twisted AdS bubble solution

After the double Wick rotation $t \to i\chi$, $\chi \to it$ and $l \to -il$ of the rotating black 3-brane solution [53 – 55], the solution looks like

$$ds^2 = \frac{1}{\sqrt{f}}(-dt^2 + h\, d\chi^2 + dx_1^2 + dx_2^2) + \sqrt{f}\left[\frac{dr^2}{\tilde{h}} - \frac{2lr_0^4\cosh\alpha}{r^4\Delta f}\sin^2\theta d\chi d\phi \right.$$
$$\left. + r^2(\Delta d\theta^2 + \tilde{\Delta}\sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2)\right], \quad (3.1)$$

where f, h, $\tilde{h}$, Δ and $\tilde{\Delta}$ are defined as follows

$$f = 1 + \frac{r_0^4\sinh^2\alpha}{r^4\Delta}, \quad \Delta = 1 - \frac{l^2\cos^2\theta}{r^2}, \quad \tilde{\Delta} = 1 - \frac{l^2}{r^2} - \frac{r_0^4l^2\sin^2\theta}{r^6\Delta f},$$

$$h = 1 - \frac{r_0^4}{r^4\Delta}, \qquad \tilde{h} = \frac{1 - \frac{l^2}{r^2} - \frac{r_0^4}{r^4}}{\Delta}. \tag{3.2}$$

The parameter l before the double Wick rotation is proportional to the angular momentum of the black brane solution.

The allowed lowest value r_H of r is given by the solution to $\tilde{h}(r) = 0$

$$r_H^2 = \frac{l^2}{2} + \sqrt{r_0^4 + \frac{l^4}{4}} \quad (> l^2). \tag{3.3}$$

It is easy to see that f, h, $\tilde{h}$, Δ and $\tilde{\Delta}$ are all positive when $r > r_H$. The total 3-brane RR-charge is proportional to $r_0^4\cosh\alpha\sinh\alpha(\propto N)$ and it is taken to be a finite constant.

If we set the angular momentum to zero $l = 0$, then it is reduced to the previous example of the AdS bubble. In the near horizon limit, we can approximate it as $f \simeq \frac{r_0^4\sinh^2\alpha}{r^4\Delta}$ and thus the AdS radius R is given by $R^2 = r_0^2\sinh\alpha$. Thus we call this solution the twisted AdS bubble.

As in the previous example we need to be careful about the regularity of the solution. The non-trivial constraint comes from the behavior near the point $\theta = 0$ and $r = r_H$.

---

[12]Refer to [56] for the time-dependent bubble solution obtained by another double Wick rotation of R-charged solutions in 5D gauged supergravity.

Around that point, the relevant part of the metric looks[13]

$$ds^2 \simeq \frac{\beta r_H^4}{r_0^4 \cosh \alpha}(r - r_H)d\chi^2 + \frac{r_0^4 \cosh \alpha}{\beta r_H^4}\frac{dr^2}{r - r_H}$$
$$+ \frac{r_0^4 \cosh \alpha}{r_H^2}\left[d\theta^2 + \theta^2(d\phi - \frac{lr_H^2}{r_0^4 \cosh \alpha}d\chi)^2\right], \qquad (3.4)$$

where $\beta = \frac{4}{r_H} - \frac{2l^2}{r_H^3}$.

The regularity requires the following two identifications

$$(\chi, \phi) \sim (\chi, \phi + 2\pi), \quad (\chi, \phi) \sim (\chi + L, \phi + 2\pi\zeta), \qquad (3.5)$$

where

$$L = \frac{2\pi r_0^4 \cosh \alpha}{2r_H^3 - l^2 r_H}, \quad \zeta = \frac{lr_H^2}{2r_H^3 - l^2 r_H}. \qquad (3.6)$$

3.2 Twisted circle background

The second condition in (3.5) looks non-trivial. Though in the asymptotic region $r \to \infty$, the form of the metric approaches the flat metric

$$ds^2 = -dt^2 + d\chi^2 + dx_1^2 + dx_2^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\Omega_3^2), \qquad (3.7)$$

the second periodicity requires that (χ, ϕ) is identified $(\chi, \phi) \sim (\chi + L, \phi + 2\pi\zeta)$. This means that the asymptotic geometry is the twisted circle (or Melvin background). The string theory on such a background was first studied in [14].

The parameter ζ which measures the strength of the twist takes the values within $0 \le \zeta < 1$. This is clear if we rewrite it as follows

$$\zeta = \sqrt{\frac{x^4 + x^2\sqrt{4 + x^4}}{2(x^4 + 4)}}, \quad \left(x \equiv \frac{l}{r_0}\right). \qquad (3.8)$$

The upper bound $\zeta < 1$ is very natural since the point $\zeta = 1$ corresponds to the supersymmetric compactification in the asymptotic region and thus there should be no bubble solution. Indeed we can see that the limit $\zeta \to 1$ is equivalent to the extremal D3-branes $l, r_0 \to 0$, keeping L and N finite.

Sometimes it is useful to define the new angular coordinate $\tilde{\phi} = \phi - q\chi$, where $qL = 2\pi\zeta$, and rewrite the metric (3.7) as follows

$$ds^2 = -dt^2 + d\chi^2 + dx_1^2 + dx_2^2 + dy^2 + y^2(d\tilde{\phi} + qd\chi)^2 + \sum_{i=1}^4 dz_i^2, \qquad (3.9)$$

where we used the transverse coordinates defined by $y \equiv r \sin \theta$ and $z_i \equiv r \cos \theta (\Omega_3)_i$. Notice that in this new coordinate system, the periodicity of $\tilde{\phi}$ and χ are are given by the ordinary (untwisted) ones $\tilde{\phi} \sim \tilde{\phi} + 2\pi$ and $\chi \sim \chi + L$.

---

[13]Notice $\Delta = \tilde{\Delta} = \frac{r_0^4}{r_H^4}$ and $f = \cosh^2 \alpha$.

The string theory on the twisted circle does not change continuously with respect to the parameter ζ. Especially, when ζ is irrational, it is known that the string theory behaves rather unusually [17, 57]. Thus below we will mainly assume that ζ takes rational values

$$\zeta = \frac{k}{M}, \tag{3.10}$$

where k and M are coprime positive integers. In this case the string theory on (3.9) is equivalent to the one on the Z_M orbifold $(R^2 \times S^1)/Z_M \times R^{1,6}$. To be more precise, the background before the Z_M projection is considered to be the ordinary supersymmetric type II string when $k + M$ is even. On the other hand, when $k + M$ is odd, it is the type II string with an antiperiodic boundary condition for fermions in the circle direction.[14] If we take the small radius limit $L \to 0$, the background becomes equivalent to the type II string (or type 0 string) on $R \times (C/Z_M)$ when $k + M$ is even (or is odd) [17].

3.3 Closed string tachyon condensation

It is known that in the twisted circle background (or Melvin background) of type II string, a closed string tachyon appears when the radius of circle is enough small [14] (string scale). This tachyon is localized near the origin of R^2 [14 – 18, 22]. Suppose N D3-branes are located at $r = 0$ of the twisted circle (3.7). Then its near horizon metric is given by

$$ds^2 = R^2 \frac{dr^2}{r^2} + \frac{r^2}{R^2}(-dt^2 + d\chi^2 + dx_1^2 + dx_2^2) + R^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2). \tag{3.11}$$

with the identification $(\chi, \phi) \sim (\chi + L, \phi + 2\pi\zeta)$. Then we find that the radius of the twisted circle χ becomes small in the IR region $r \ll 1$. Thus we expect the closed string tachyon condensation in that region. We would like to argue that the end point is given by the twisted bubble (3.1) in the similar sense of the previous example of the AdS bubble. To make this argument clearer, we can assume the shell distribution of the D3-branes so that the flat spacetime is realized inside the shell as before. We will presents non-trivial evidences by comparing the energy density and the entanglement entropy between the gauge and gravity side in the following subsections.

To see if this speculation makes sense, it is useful to see how the twisted boundary condition (3.5) at the UV boundary $r = \infty$ evolves toward the IR region $r \to r_0$. If we rewrite the metric (3.1) near $\theta = 0$ in the form $A(d\chi)^2 + B\theta^2(d\phi - q(r)d\chi)^2 + \cdots$, the twist parameter $q(r)$ is given by

$$q(r) = \frac{lr_0^4 \cosh\alpha}{\Delta^2 f r^6}. \tag{3.12}$$

This becomes monotonically large toward the IR region and it becomes zero in the UV limit $r = \infty$. Since we put the twisted boundary condition at $r = \infty$ and the non-zero value of $q(r)$ cancels the effect of the twist, the strength of the twist becomes weaker as we go into the IR region and it vanishes at $r = r_H$ smoothly. This is qualitatively consistent

---

[14]In other words, it is the Z_2 orbifold of type II string by the action $\sigma_{1/2} \cdot (-1)^{F_S}$, where $\sigma_{1/2}$ is the half shift along the circle and F_S is the spacetime fermion number.

with the fact that the closed string tachyon condenses only in the IR region, where the radius of the circle becomes stringy size and that the UV geometry should not change.

At the same time, another important property of the closed string tachyon in the AdS twisted circle is that it is localized near the S^3 ('north pole') defined by $\theta = 0$ within the whole S^5. This is the crucial difference between this example and the previous one in section 2. Indeed, the geometry depends on the position of S^5 as is clear from the metric (3.1). We can find that the radius of the twisted circle (defined by the shift $\Delta\chi = L$, $\Delta\phi = 2\pi\zeta$) depends on θ and it is non-zero except the north pole S^3. Thus the twisted circle shrinks to zero size in the north pole. This is consistent with the fact that the tachyon is localized at the north pole and the observation in [7] that the winding tachyon will pinch off the wound circle.

3.4 Asymptotically flat solution with a conical singularity

In the previous subsection the background asymptotically approaches the twisted circle and thus is not asymptotically flat. Instead, we can consider the same solution (3.1) with requiring the asymptotic flatness. Inevitably, we will encounter conical singularities in the IR region at $\theta = 0, r = r_H$. Notice that we consider this asymptotically flat solution only in this subsection among all parts of the present paper.

Let us first examine what types of the singularities appear in the IR region. The asymptotic flatness requires the coordinates ϕ and χ are compactified in a usual way i.e.

$$(\chi, \phi) \sim (\chi + L', \phi), \qquad (\chi, \phi) \sim (\chi, \phi + 2\pi), \tag{3.13}$$

where the periodicity L' is not necessarily equal to L in (3.6). Now we assume the combination $\frac{\zeta L'}{L}$ is a rational number and we express it as $\frac{\zeta L'}{L} = \frac{k}{M}$, where M and k are coprime integers. Define the following two angles

$$\tilde{\chi} \equiv \frac{2\pi\zeta}{kL}\chi, \quad \tilde{\phi} = \phi - \frac{2\pi\zeta}{L}\chi. \tag{3.14}$$

They satisfy the following periodicity

$$(\tilde{\chi}, \tilde{\phi}) \sim (\tilde{\chi} + \frac{2\pi}{M}, \tilde{\phi} - \frac{2\pi k}{M}), \qquad (\tilde{\chi}, \tilde{\phi}) \sim (\tilde{\chi}, \tilde{\phi} + 2\pi). \tag{3.15}$$

Thus if we define the following coordinate of $R^4 = C^2$ in the neighborhood of $\theta = 0, r = r_*$ (irrelevant constant factors are denoted by a and b)

$$Z_1 = a\sqrt{r - r_*}e^{i\tilde{\chi}}, \quad Z_2 = b\theta e^{i\tilde{\phi}}, \tag{3.16}$$

the singular geometry is described by the orbifold C^2/Z_M described by the Z_M action

$$(Z_1, Z_2) \sim (Z_1 e^{\frac{2\pi i}{M}}, Z_2 e^{-\frac{2\pi ik}{M}}). \tag{3.17}$$

Next we would like to estimate the energy of these configurations. It is known that the ADM energy of the rotating black hole is given by the same formula as the non-rotating one i.e. $l = 0$ [54, 53]. This is because its asymptotic geometry $r \to \infty$, where we read off

the ADM mass, does not depend on the parameter l [58, 41]. The same is true for our case since the double Wick rotation does not essentially touch[15] the terms which depend on l.

Thus we obtain the energy density of the twisted bubble

$$T_{00} = \frac{\pi^2 r_0^4}{16 G_N^{(10)}} (1 + 4 \sinh^2 \alpha). \tag{3.18}$$

We would like to subtract the energy stress tensor of extremal D3-branes from (3.18). The extremal limit is given by $r_0 \to 0$ (or $\alpha \to \infty$). To do this subtraction we should keep the total RR-flux same in both sides. The RR-flux is proportional to $r_0^4 \sinh \alpha \cosh \alpha$ as we mentioned. The energy density of extremal D3-branes with the same amount of RR-flux is

$$T_{00}^{(0)} = \frac{\pi^2 r_0^4}{4 G_N^{(10)}} \sinh \alpha \cosh \alpha. \tag{3.19}$$

After the subtraction the energy density becomes

$$T_{00} - T_{00}^{(0)} = -\frac{\pi^2 r_0^4}{16 G_N^{(10)}}. \tag{3.20}$$

Even though this expression looks equivalent to the AdS bubble ($l = 0$), its physical value depends on l non-trivially via the relation (3.6) (we always fix the value of L).

3.5 Casimir energy

Now we come back to the geometry (3.1) with the twisted identification (3.5). It is dual to the SU(N) Yang-Mills theory with the twisted boundary condition. This originates from the D3-branes wrapped on the circle S^1 in the orbifold $(R^2 \times S^1)/Z_M$ [20, 3] (see also [21]). Let us compute the Casimir energy in this gauge theory.

The transverse (complex) scalar in the R^2 direction is denoted by Φ, and the other scalars are denoted by ϕ. Their twisted boundary conditions are written as

$$\Phi_{ab}(z + L) = e^{\frac{2\pi i}{M}(a-b+k)} \Phi_{ab}(z), \qquad \phi_{ab}(z + L) = e^{\frac{2\pi i}{M}(a-b)} \phi_{ab}(z), \tag{3.21}$$

where a and b represents the Wilson line in the circle direction and take values[16] $0 \le a, b < M$. For the fermions we similarly find

$$\psi_{ab}(z + L) = e^{\frac{\pi i(k+M)}{M}} e^{\frac{2\pi i(a-b)}{M}} \psi_{ab}(z). \tag{3.22}$$

The two point functions for these fields can be found easily. For example, the one for the field ϕ becomes

$$\langle \phi(x)_{ab} \phi(x')_{ba} \rangle = \frac{1}{4\pi^2} \sum_{n \in \mathbf{Z}} \frac{e^{\frac{2\pi i}{M}(a-b)n}}{(x-x')^2 + (y-y')^2 + (z-z'-nL)^2 - (t-t')^2}. \tag{3.23}$$

---

[15]Even though the off diagonal term $\propto dt d\phi$ depends on l, its coefficient becomes too small to contribute to the ADM mass when r is large.

[16]As shown in [20], the values $i \le a, b < i+1$ corresponds to the i-th fractional branes in the orbifold limit $L \to 0$.

As in section 2.2, it is straightforward to compute the Casimir energy from (3.23). When we consider N D3-branes with the same value of the Wilson line, we obtain

$$
\begin{aligned}
T_{00} &= \frac{N^2}{\pi^2 L^4}\left[-6\sum_{n=1}^{\infty}\frac{1}{n^4} - 2\sum_{n=1}^{\infty}\frac{\cos(\frac{2\pi nk}{M})}{n^4} + 8\sum_{n=1}^{\infty}\frac{\cos(\frac{\pi n(k+M)}{M})}{n^4}\right] \\
&= -\frac{N^2\pi^2}{L^4}\left[\frac{1}{6} - \frac{k^2}{M^2} + \frac{4k^3}{3M^3} - \frac{k^4}{2M^4}\right] \\
&= \frac{N^2\pi^2}{L^4}\left[\frac{1}{6} - \zeta^2 + \frac{4}{3}\zeta^3 - \frac{1}{2}\zeta^4\right],
\end{aligned}
\tag{3.24}
$$

where we have employed the identity

$$
\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^4} = \frac{1}{48}\left[2\pi^2(x-\pi)^2 - (x-\pi)^4 - \frac{7}{15}\pi^4\right].
\tag{3.25}
$$

This energy density (3.24) is a monotonically increasing function of ζ (see the lower graph in figure 3). In particular, it takes the vanishing value $T_{00} = 0$ at the supersymmetric point $\zeta = 1$ and the previous value (2.12) at $\zeta = 0$. Notice also that T_{00} is negative except the supersymmetric point, which is consistent with our claim that the closed string tachyon condensation leads to the twisted AdS bubble.

In the above we assumed that D3-branes at the twisted circle $(R^2 \times S^1)/Z_M$ have the same value of the Wilson line a. In the orbifold theoretic language, such branes are called fractional D3-branes of the same type. We cannot move them away from the origin of R^2 without exciting the system [20, 19].

It is also intriguing to consider a bulk D3-brane, which is equivalent to a linear combination of M fractional D3-branes of different types. A bulk D3-brane has a moduli which shifts its position away from the origin. To compute the Casimir energy of N bulk D3-branes we need to sum over a and b such that $a, b = 0, 1, 2, \cdots, M-1$. This can be easily done because $\sum_{a,b} e^{\frac{2\pi i}{M}(a-b)n} = M^2 \cdot \delta_{n,M\mathbf{Z}}$. Thus the total Casimir energy is given by replacing L with ML and multiplying M^2. Also we have to be careful about the boundary condition for fermions. In the end we find the following result: when $k + M$ is even, the energy is vanishing, while $k + M$ is odd, it is given by

$$
T_{00} = -\frac{\pi^2 N^2}{6M^2 L^4}.
\tag{3.26}
$$

It is now clear that the system of bulk D3-branes has a larger energy compared with that of the fractional D3-branes and thus it is unstable. These results for the bulk D3-branes also tell us that the dual background is given by the Z_M orbifold of the pure AdS or of the (untwisted) AdS bubble with the periodicity $\chi \sim \chi + ML$ when $k + M$ is even or odd, respectively. Thus when the $k + M$ is odd, the bulk tachyon is condensed in the IR region, while $k + M$ is even, closed string tachyons is not condensed.

In other words, the tachyon condensation process from the AdS twisted circle to the AdS twisted bubble is dual to the shift of the Wilson line expectation values[17] from those for the bulk D3-branes to those for the fractional D3-branes if $k + M$ is even.

To make the above point clearer, let us compare the energy computed in the free Yang-Mills with the one (3.20) found in the gravity side. Here we have to be careful since we are treating the energy in a background which is not asymptotically flat or AdS. Nevertheless, we assume the result (3.20), which was obtained by requiring the asymptotic flatness, is also true for our case with the twisted boundary condition. This is reasonable because the twisted circle is a freely acting orbifold and will not produce any extra energy as opposed to the conical orbifold C/Z_n [40, 39].

The energy density in the gravity side reads in terms of the gauge theoretic variables

$$T_{00} = -\frac{\pi^2 N^2}{8L^4} \cdot \frac{1}{(x^2/2 + \sqrt{1 + x^4/4})^2 (1 + x^4/4)^2} \simeq -\frac{\pi^2 N^2}{L^4} \left(\frac{1}{8} - \frac{1}{2}\zeta^2 + \cdots\right), \quad (3.27)$$

where in the final expression we wrote down the power expansion with respect to ζ. This result is plotted as a upper graph in figure 3. It takes the values $T_{00} \to -\frac{\pi^2 N^2}{8L^4}$ ($\zeta \to 0$) and $T_{00} \to 0$ ($\zeta \to 1$). The qualitative behavior of the gravitational energy agrees with the Casimir energy in free Yang-Mills (3.24) rather successfully as is clear from figure 3. The ratio of the energy density in both sides is given by $\frac{T_{00}^{\text{freeYM}}}{T_{00}^{\text{gravity}}} = \frac{4}{3}$ for the AdS bubble $\zeta = 0$ as we have already reviewed in section 2.2. A new result here is that in the extremal limit $\zeta \to 1$ it approaches

$$\frac{T_{00}^{\text{freeYM}}}{T_{00}^{\text{gravity}}} = \frac{9}{8}. \quad (3.28)$$

This value is closer to 1 than the result at $\zeta = 0$, which is very natural because the free Yang-Mills can be a better approximation to the strongly coupled Yang-Mills in the almost BPS case than in the deeply non-BPS case.

3.6 Entanglement entropy

The entanglement entropy in the free $U(N)$ N=4 Yang-Mills theory can be obtained as in section 2.4. The result for the field with the twisted boundary condition $\phi(z + 2\pi R) = e^{\frac{2\pi i}{M}a}\phi(z)$ is obtained from the untwisted one by replacing the sum $\sum_{q \in Z} e^{-\frac{\pi^2 R^2 q^2}{s}}$ with $\sum_{q \in Z} e^{\frac{2\pi i}{M}aq} \cdot e^{-\frac{\pi^2 R^2 q^2}{s}}$ in (2.30). We can perform the summation by using the formula

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{1}{4}(x - \pi)^2 - \frac{\pi^2}{12}. \quad (3.29)$$

Suppose N fractional D3-branes on the twisted circle. The entropy can be found as

$$S_A = 12N^2 \cdot S_A^{\text{area law}} + \frac{N^2 \pi V_1}{6L} \left(1 + \frac{3k^2}{M^2} - \frac{2k}{M}\right), \quad (3.30)$$

where $S_A^{\text{area law}}$ is defined by (2.32). Thus we obtain

$$\Delta S_A^{\text{freeYM}} = -\frac{N^2 \pi V_1}{6L} \left(1 + 2\zeta - 3\zeta^2\right). \quad (3.31)$$

---

[17]This means that the tachyon condensation corresponds to the clumped eigenvalues of Wilson loop. This looks analogous to the behavior of the 2D maximally supersymmetric Yang-Mills pointed out in the paper [59], which relates the clumping phenomena to the Gregory-Laflamme black-hole/black string transition. It is also similar to the free Yang-Mills analysis [60, 61] of the deconfinement phase transition [36].

Since this is clearly negative $\Delta S_A < 0$ (for $0 \leq \zeta < 1$), we again confirm that the entanglement entropy decreases under the closed string tachyon condensation.
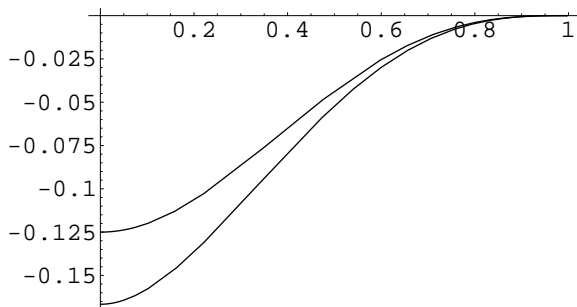

Figure 3: The Casimir energy (in the normalization of $\frac{L^4}{\pi^2 N^2} \cdot T_{00}$) as a function of the twist parameter $0 \leq \zeta \leq 1$ is presented in both gravity and free Yang-Mills side. The one starts with the value -0.125 is the gravity result and the other is the free Yang-Mills result.

In the case of N bulk D3-branes, the entropy takes the following form (we assume $k + M$ is odd)

$$S_A = 12M^2 N^2 \cdot S_A^{\text{area law}} + \frac{N^2 \pi V_1}{6L}, \tag{3.32}$$

where $S_A^{\text{area law}}$ is defined by (2.32). Thus the finite term does not depend on M. When $k + M$ is even, the second finite term is given by $\frac{N^2 \pi V_1}{3L}$. Again these results are consistent with the previous claim that the dual background of the bulk 3-branes is given by the orbifold of the pure AdS or of the (untwisted) AdS bubble with the periodicity ML when $k + M$ is even or odd,

respectively, by considering the entropy density $\frac{\Delta S_A}{LV_1}$.

Next we compare these with the gravity calculation. We would like to apply the holographic computation of the entanglement entropy to the near horizon limit $e^\alpha \gg 1$ of the twisted bubble (3.1). In this example, the total 10D spacetime is relevant and thus we need to apply the holographic formula generalized into ten dimension [11]

$$S_A = \frac{1}{4G_N^{(10)}} \int_{\gamma_A \times S^5} \sqrt{g}, \tag{3.33}$$

which was already employed in (2.17). We only consider the simplest case where the subsystem is defined by dividing the total space into half parts as in (2.14). After some algebras we find that the integral in (3.33) becomes drastically simplified as

$$S_A = \frac{V_1 L}{4G_N^{(5)}} \int_{r_H}^{r_\infty} dr \frac{r}{R} = \frac{V_1 L}{4G_N^{(5)} R} \left(\frac{r_\infty^2}{2} - \frac{r_H^2}{2} \right). \tag{3.34}$$

In terms of the gauge theoretic language assuming $\zeta = \frac{k}{M}$, the second finite term is equivalently rewritten as ($x = l/r_0$)

$$\Delta S_A^{\text{gravity}} = -\frac{\pi N^2 V_1}{(4 + x^4)L}. \tag{3.35}$$

Again we confirmed $\Delta S_A < 0$ and this agrees with our conjecture.

The figure 4 summarizes the results in both free Yang-Mills (upper) and gravity side (lower). First we notice that in the near extremal region $1 - \zeta \ll 1$, the entropy coincides precisely

$$\Delta S_A^{\text{freeYM}} \simeq \Delta S_A^{\text{gravity}} \simeq \frac{2\pi N^2 V_1}{3L}(\zeta - 1). \tag{3.36}$$
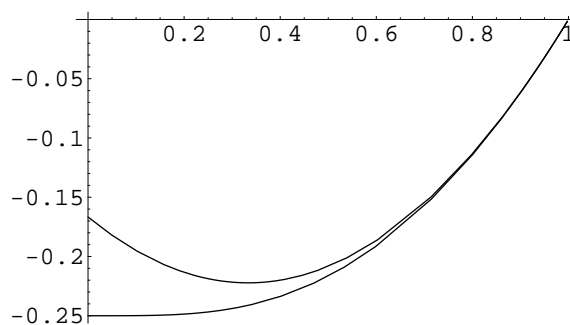
Figure 4: The entanglement entropy (in the normalization of $\frac{L}{\pi N^2 V_1} \Delta S_A$) as a function of the twist parameter $0 \leq \zeta \leq 1$ is presented in both gravity and free Yang-Mills side. The one starts with the value -0.25 is the gravity result and the other is the free Yang-Mills one.

On the other hand, near $\zeta = 0$, their behaviors are slightly different. The free Yang-Mills entropy takes a minimum value at $\zeta = \frac{1}{3}$, while the gravity (or strongly coupled gauge theory) entropy is a monotonically increasing function. Since the entanglement entropy is not protected by supersymmetries, we may have to be satisfied with this result, though it is not clear why the free Yang-Mills entropy takes the minimum value. The quantitative agreement in the near extremal region (3.36) is remarkable from this conventional viewpoint and may be regarded as a further evidence for AdS/CFT correspondence in slightly non-BPS backgrounds.

3.7 More general solutions

We can construct more general twisted AdS bubble solutions by the double Wick rotation of rotating D3-brane solutions [55] with three angular momenta (or equally three R-charges)

$$ds^2 = \frac{1}{\sqrt{f}}(-dt^2 + h\, d\chi^2 + dx_1^2 + dx_2^2) + \sqrt{f}\Bigg[\frac{dr^2}{\tilde{h}} + r^2 \sum_{i=1}^{3} H_i(d\mu_i^2 + \mu_i^2 d\phi_i^2)$$

$$-\frac{2r_0^4 \cosh\alpha}{r^4 \Delta f} d\chi \Bigg(\sum_{i=1}^{3} l_i \mu_i^2 d\phi_i\Bigg) - \frac{r_0^4}{r^4 \Delta f}\Bigg(\sum_{i=1}^{3} l_i \mu_i^2 d\phi_i\Bigg)^2\Bigg], \tag{3.37}$$

where f, h, $\tilde{h}$, Δ and $H_i (i = 1, 2, 3)$ are defined as follows

$$f = 1 + \frac{r_0^4 \sinh^2\alpha}{r^4 \Delta}, \quad H_i = 1 - \frac{l_i^2}{r^2}, \quad \Delta = H_1 H_2 H_3 \sum_{i=1}^{3} \frac{\mu_i^2}{H_i},$$

$$h = 1 - \frac{r_0^4}{r^4 \Delta}, \quad \tilde{h} = \frac{H_1 H_2 H_3 - \frac{r_0^4}{r^4}}{\Delta}, \tag{3.38}$$

and

$$(\mu_1, \mu_2, \mu_3) = (\sin\theta, \cos\theta\sin\phi, \cos\theta\cos\phi). \tag{3.39}$$

If we set $l_2 = l_3 = 0$, this metric is reduced to the previous example with the one angular momentum, identifying $l = l_1$. Here we consider the two angular momentum case setting $l_3 = 0$ just for simplicity.

The lower limit r_H is given $r_H^2 = \frac{l_1^2 + l_2^2}{2} + \sqrt{r_0^4 + \frac{(l_1^2 - l_2^2)^2}{4}}$, and f, h, $\tilde{h}$, Δ and $H_i(i = 1, 2, 3)$ are all positive when $r > r_H$. Around the point $\theta = \phi = 0$ and $r = r_H$, the metric looks like

$$
\begin{aligned}
ds^2 \simeq & \frac{\beta r_H^4}{r_0^4 \cosh \alpha}(r - r_H)d\chi^2 + \frac{r_0^4 \cosh \alpha}{\beta r_H^4} \frac{dr^2}{r - r_H} \\
& + (r_H^2 - l_1^2) \cosh \alpha \left[d\theta^2 + \theta^2 (d\phi_1 - \frac{l_1}{(r_H^2 - l_1^2) \cosh \alpha} d\chi)^2 \right] \\
& + (r_H^2 - l_2^2) \cosh \alpha \left[d\phi^2 + \phi^2 (d\phi_2 - \frac{l_2}{(r_H^2 - l_2^2) \cosh \alpha} d\chi)^2 \right],
\end{aligned}
\tag{3.40}
$$

where $\beta = \frac{4}{r_H} - \frac{2(l_1^2 + l_2^2)}{r_H^3}$.

The regularity requires the following three identifications

$$
\begin{aligned}
(\chi, \phi_i) & \sim (\chi, \phi_i + 2\pi), \quad (i = 1, 2) \\
(\chi, \phi_1, \phi_2) & \sim (\chi + L, \phi_1 + 2\pi\zeta_1, \phi_2 + 2\pi\zeta_2),
\end{aligned}
\tag{3.41}
$$

where

$$
L = \frac{2\pi r_0^4 \cosh \alpha}{2r_H^3 - (l_1^2 + l_2^2)r_H}, \quad \zeta_i = \frac{l_i r_0^4}{(2r_H^3 - (l_1^2 + l_2^2)r_H)(r_H^2 - l_i^2)}.
\tag{3.42}
$$

The parameters ζ_1, ζ_2 take the values within $0 \leq \zeta_1, \zeta_2 < 1$ and $0 \leq \zeta_1 + \zeta_2 < 1$. Suppose that the twist parameters ζ_1, ζ_2 take rational values

$$
\zeta_i = \frac{k_i}{M}, \quad (i = 1, 2),
\tag{3.43}
$$

where k_i and M are coprime positive integers. Then the string theory approaches the one defined on the (generalized) twisted circle $(C^2 \times S^1)/Z_M \times R^{1,4}$ [17, 18] in the asymptotic region $r \to \infty$. Since the point $\zeta_1 = \zeta_2 = 0$ corresponds to the anti-periodic boundary condition for fermions, the sixteen supersymmetries are preserved when the following condition is satisfied [17, 18]

$$
\zeta_1 + \zeta_2 = 1.
\tag{3.44}
$$

In the dual gauge theory side, the Yang-Mills theory becomes $N = 2$ supersymmetric. Except these supersymmetric points, the string theory on the twisted circle includes tachyon field as before [17, 18]. We wish to claim that in the presence of D3-branes the endpoint of the closed string tachyon condensation is given by the twisted AdS bubble solution (3.37).

Let us compute the Casimir energy. The transverse (complex) scalar in the C^2 direction is denoted by Φ^i ($i = 1, 2$), and the other scalars are denoted by ϕ. Their twisted boundary conditions are written as

$$
\Phi_{ab}^i(z + L) = e^{\frac{2\pi i}{M}(a - b + k_i)} \Phi_{ab}^i(z), \quad \phi_{ab}(z + L) = e^{\frac{2\pi i}{M}(a - b)} \phi_{ab}(z),
\tag{3.45}
$$

where $0 \leq a, b < M$. For the fermions we similarly find

$$
\psi_{ab}^\pm(z + L) = e^{\frac{\pi i(k_1 \pm k_2 + M)}{M}} e^{\frac{2\pi i(a - b)}{M}} \psi_{ab}^\pm(z).
\tag{3.46}
$$

When we consider N (fractional) D3-branes with the same value of the Wilson line, we obtain

$$T_{00}^{\text{freeYM}} = \frac{N^2}{\pi^2 L^4}\left[-4\sum_{n=1}^{\infty}\frac{1}{n^4} - 2\sum_{n=1}^{\infty}\frac{\cos(\frac{2\pi n k_1}{M}) + \cos(\frac{2\pi n k_2}{M})}{n^4}\right.$$

$$\left.+4\sum_{n=1}^{\infty}\frac{\cos(\frac{\pi n(k_1+k_2+M)}{M}) + \cos(\frac{\pi n(k_1-k_2+M)}{M})}{n^4}\right]$$

$$= -\frac{N^2\pi^2}{L^4}\left[\frac{1}{6} - (\zeta_1^2 + \zeta_2^2) + \frac{4}{3}(\zeta_1^3 + \zeta_2^3) - \frac{1}{2}(\zeta_1^2 - \zeta_2^2)^2\right]. \tag{3.47}$$

On the other hand, in the gravity side, the energy density of this twisted bubble is given

$$T_{00}^{\text{gravity}} = -\frac{\pi^2 r_0^4}{16 G_N^{(10)}} = -\frac{\pi^2 N^2}{8L^4}\cdot\left(1 + \frac{(x^2-y^2)^2}{4}\right)^{-2}\cdot\left(\frac{x^2+y^2}{2} + \sqrt{1 + \frac{(x^2-y^2)^2}{4}}\right)^{-2},$$
$$\tag{3.48}$$

where $x = l_1/r_0$ and $y = l_2/r_0$. We can again confirm the qualitative agreement of the behavior of the energy density between the free Yang-Mills and gravity result. We can check in both sides that T_{00} vanishes along the supersymmetric points (3.44) as expected.

We can also examine the entanglement entropy as in the previous case. The gravity computation leads to

$$\Delta S_A^{\text{gravity}} = -\frac{\pi N^2 V_1}{(4 + (x^2-y^2)^2)L}. \tag{3.49}$$

On the other hand, the free Yang-Mills result reads

$$\Delta S_A^{\text{freeYM}} = -\frac{\pi N^2 V_1}{6L}\left(1 + 2(\zeta_1 + \zeta_2) - 3(\zeta_1^2 + \zeta_2^2)\right). \tag{3.50}$$

We can check the qualitative agreement between them as before. The limits which approach backgrounds with sixteen supersymmetries (i.e. $\zeta_1 + \zeta_2 = 1$ as in (3.44)) are given by

$$x^2 - y^2 = \alpha^2, \quad x \to \infty, \quad y \to \infty, \tag{3.51}$$

where α is a finite constant. Then the twist parameters are given by

$$\zeta_1 = \frac{\frac{\alpha^2}{2} + \sqrt{1 + \frac{\alpha^4}{4}}}{2\sqrt{1 + \frac{\alpha^4}{4}}}, \qquad \zeta_2 = \frac{-\frac{\alpha^2}{2} + \sqrt{1 + \frac{\alpha^4}{4}}}{2\sqrt{1 + \frac{\alpha^4}{4}}}. \tag{3.52}$$

In this limit, the free Yang-Mills result (3.50) is simplified as follows

$$\Delta S_A^{\text{freeYM}} \to -\frac{\pi N^2 V_1}{L}\cdot\frac{1}{4 + \alpha^4}. \tag{3.53}$$

This precisely agrees with the gravity result (3.49). In this comparison, the important point is that the entropy for the $N = 2$ super Yang-Mills is different from that for the $N = 4$ super Yang-Mills. We can also regard this successful quantitative agreement as a further support for the assumed holographic calculation (3.33). It will be a moderate exercise to extend the above results to the three parameter cases $l_i \neq 0$ $(i = 1, 2, 3)$, which include $N = 1$ super Yang-Mills theories.

3.8 Comparison with known results from world-sheet RG-flow analysis

It will also be helpful to compare[18] the above decay process with the results obtained from the world-sheet RG-flow [22, 62] (for a review see [3]) using the gauged linear σ-model. In this analysis, the decay of the twisted circle in flat space (i.e. $(R^2 \times S^1)/Z_M$) is considered and thus there are no D3-branes. The vanishing of the twisted circle S^1 is also observed in this world-sheet RG-flow analysis [62]. However, after the circle vanishes, another circle (called the supersymmetric cycle) $\tilde{S}^1$ appears and the endpoint becomes $R^2 \times \tilde{S}^1$, where the radius of the new circle becomes M times that of the twisted circle [22, 15]. This latter process is not included in our AdS counter part (3.1).

Probably, this difference is due to the presence of the cosmological constant or equally of the D3-branes. Since the conservation of the twisted sector RR-charges is violated by the closed string tachyon condensation [62] (see also [63] for the two dimensional orbifold C/Z_N), we expect a large back reaction in the presence of fractional D3-branes. Indeed, as we have seen, the D-branes which constitute the AdS twisted bubble (3.1) are identified with the fractional D3-branes of the same kind. It will be an interesting future problem to explore this issue.

4. Null boundaries in string theory and closed string tachyons

Up to now, we have discussed static bubble solutions in string theory.[19] Generally, such a background is described by a complicated metric and RR flux and it is not easy to solve the corresponding string theory. One way to simplify the background is to take a particular limit without ending up with a trivial solution. Consider an infinite boost of the asymptotically flat bubble solution (3.1). Remember that in the previous section we claimed that this background (3.1) is an end point of the closed string tachyon condensation. In particular, we set $l = 0$ in (3.1) for simplicity. This is the static bubble solution whose near horizon limit is the AdS bubble (2.4).

After the infinite boost $t \pm r \to \gamma^{\mp}(t \pm r)$ and $\gamma \to \infty$, we find that the metric becomes simplified as follows

$$ds^2 \simeq -dt^2 + dr^2 + d\chi^2 + dx_1^2 + dx_2^2 + \frac{\gamma^2}{4}(t-r)^2 d\Omega_5^2$$

$$\to -dt^2 + dr^2 + d\chi^2 + dx_1^2 + dx_2^2 + (t-r)^2 \sum_{i=1}^{5} dy_i^2, \tag{4.1}$$

where the final expression can be found by noting that the five sphere can be approximated by $y^i \in R^5$ in the limit $\gamma \to \infty$. Furthermore, since the original radial coordinate is restricted to the values $r \ge r_0$, the allowed values of (t, r) in (4.1) become after the boost

$$r - t > 0. \tag{4.2}$$

---

[18]We are very grateful to Takao Suyama for useful discussions about the materials in this subsection.

[19]Most of the contents in this section can be read without using the results in previous sections. Also the readers can skip this section who are interested in the discussions of closed string tachyon condensations from the viewpoint of AdS/CFT correspondence.

Thus this spacetime has a null boundary[20] (or light-like boundary) at $r - t = 0$.

In general, there has been no systematic understanding on what kinds of spacetime boundaries are allowed in the string theory. Therefore, it will be helpful to examine various string theory backgrounds with spacetime boundaries. As we will see below this subject is closely related to the closed string tachyon condensation since the tachyon wall can be regarded as a spacetime boundary. At the same time, a non-static boundary in spacetime offers us a simple time-dependent background in string theory.

Below we will discuss exactly solvable examples with null boundaries in critical string theory. They are obtained from bulk closed string tachyon condensations[21] simpler than the quasi local tachyon condensations relevant for (2.4) and (3.1). They are so simple that their spacetimes except the null boundaries are just the ordinary 26 or 10 dimensional flat spacetime, where the perturbative description of string theory can be done exactly.

4.1 Null boundaries in bosonic string

Consider the 26 dimensional critical bosonic string. The coordinates in the bosonic string are denoted by x_μ $(\mu = 0, 1, 2, \cdots, 25)$ and their world-sheet fields are written as X_μ. We assume the null (or light-like) linear dilaton[22] in this background[23]

$$g_s = e^{-Q(x_0 + x_1)}. \tag{4.3}$$

Notice that the total central charge for the world-sheet fields X_0 and X_1 remains $c = 2$ and thus the background is still critical.

Furthermore we put the Liouville potential

$$S_L = \mu \int dz^2 e^{-2bX_1}, \tag{4.4}$$

where $b > 0$ is determined from the relation $Q = b + 1/b$. This regulates the strongly coupled regions at large X_1.

Now we perform the infinite Lorentz boost such that the linear dilaton gradient becomes zero. Explicitly, this is realized by defining the new boosted coordinates $\tilde{x}_0, \tilde{x}_1$ as follows

$$\tilde{x}_0 + \tilde{x}_1 = \gamma(x_0 + x_1), \quad \tilde{x}_0 - \tilde{x}_1 = \gamma^{-1}(x_0 - x_1), \tag{4.5}$$

and taking the limit $\gamma \to \infty$. It is trivial to see $g_s =$ const. after this limit is taken.

After this boost, the Liouville potential looks like

$$S_L = \mu \int dz^2 \, e^{b\gamma(\tilde{X}_0 - \tilde{X}_1)}. \tag{4.6}$$

---

[20]If we treat the light-cone time $t - r$ as a real time, this background describes a big crunch-like singularity with y_i compactified appropriately, which is almost the same as a half of the spacetime considered in [64].

[21]Clearly, another series of string theory backgrounds with spacetime boundaries can also be found from orbifold theories.

[22]Recently, the null linear dilaton background in the critical type II string is investigated as a model of cosmological singularity [65].

[23]We set $\alpha' = 1$ in this paper. The OPE is normalized such that $X^\mu(z)X^\nu(0) \sim \eta^{\mu\nu} \log z$.

Since we are taking the limit $\gamma \to \infty$, the potential (4.6) kills the half of the spacetime. Thus only the part

$$\tilde{x}_1 - \tilde{x}_0 > 0, \tag{4.7}$$

survives and the fields can propagate only there. In other words, the closed string tachyon T condenses completely $T \to \infty$ in $\tilde{x}_1 - \tilde{x}_0 < 0$ and that part of the spacetime disappears as in figure 5. On the other hand, the tachyon field is vanishing for the opposite region $\tilde{x}_1 - \tilde{x}_0 > 0$.
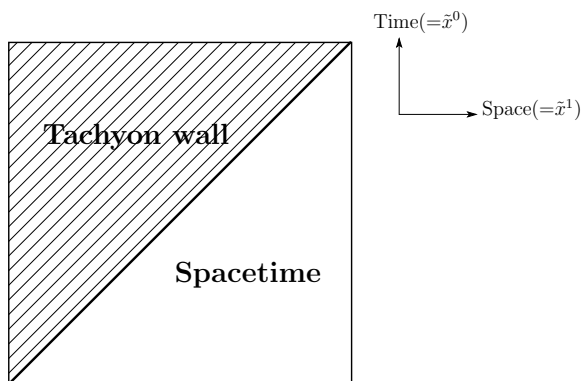

Figure 5: A spacetime with a null boundary $\tilde{x}_1 - \tilde{x}_0 = 0$ induced by a closed string tachyon condensation. The tachyon condenses completely in the shaded region and the physical spacetime is given by the opposite half region $\tilde{x}_1 - \tilde{x}_0 > 0$.

Usually the Liouville potential can be interpreted as a tachyon wall. The wall in our example after the infinite boost becomes completely rigid in that the tachyon becomes suddenly infinite when $\tilde{x}_0 - \tilde{x}_1 = 0$. The wall moves at the speed of light toward a static observer (see figure 5). We can equally obtain the opposite background (i.e. defined by $\tilde{x}_0 + \tilde{x}_1 > 0$ instead of (4.7)) by flipping the sign of x_0.

This background can be regarded as a flat space with a null boundary. This is because the dilaton and metric is trivial in the region (4.7) as is clear from the above discussion. This background will be one of the simplest examples of spacetime boundaries in the critical bosonic string theory (*cf.* analogous models [27, 66, 67, 29] in 2D string theory). These arguments can be easily generalized to the critical type 0 string theory where a similar type of closed string tachyon field exists.

4.2 Null boundaries in type II string

It is more interesting to ask if a similar null boundary in the flat space is allowed in the critical type II superstring. In this case we need to take an additional coordinate X_2 into account. In order to obtain the Liouville potential, we compactify X_2 such that we can put the $N = 2$ Liouville potential ($\Phi = X_1 + iX_2$)

$$S_L = \mu \int dz^2 d\theta^2 e^{-\frac{\Phi}{Q}} + (h.c.), \tag{4.8}$$

in the null linear dilaton background (4.3). By boosting as before, we again find that only the half of spacetime (4.7) survives. Since we can choose Q independently, we can decompactify the circle.[24] Therefore we can conclude that we can put a null boundary also in type II string theory as in figure 5.

---

[24]Remember the radius of circle is proportional to Q.

We can obtain the same result by taking T-dual in the circle direction X_2. The FZZ duality leads to the equivalent background with the following non-trivial metric and dilaton [68]

$$ds^2 = -(dx_0)^2 + (dx_1)^2 + \frac{1}{Q^2}\tanh^2(Qx_1)(d\theta)^2, \qquad g_s = \frac{e^{Qx_0}}{\cosh(Qx_1)}, \qquad (4.9)$$

where θ is compactified such that $\theta \sim \theta + 2\pi$. Also the value of x_1 is restricted as $x_1 \geq 0$.

Now we perform the previous boost (4.5). When γ is very large, $x_0 \sim x_1$ becomes large too, and thus we can approximate the string coupling as $g_s \sim e^{-Q(x_0+x_1)} = e^{-Q\gamma^{-1}(\tilde{x}_0+\tilde{x}_1)}$. The metric is also approximated by

$$ds^2 \simeq -(d\tilde{x}_0)^2 + (d\tilde{x}_1)^2 + \frac{1}{Q^2}\tanh^2\left(\frac{Q\gamma}{2}(\tilde{x}_1 - \tilde{x}_0)\right)(d\theta)^2. \qquad (4.10)$$

The restriction $x_1 \geq 0$ means

$$\tilde{x}_1 - \tilde{x}_0 > 0. \qquad (4.11)$$

Finally we take the limit $\gamma \to \infty$. Then this spacetime is identified with the flat space with a rigid wall (or boundary) at $\tilde{x}_1 - \tilde{x}_0 = 0$, as expected. Notice that this argument is analogous to our previous one (4.1).

After the infinite boost, we recover the flat type II string in the half spacetime (4.11) and thus all of the 32 supersymmetries are preserved in the bulk points.[25] However, the supersymmetry is completely broken at the rigid wall $\tilde{x}_0 + \tilde{x}_1 = 0$.

4.3 More null boundaries

In the above examples, the metric, dilaton and tachyon are the same as those in the ordinary flat spacetime except the boundaries. Thus we expect that the equation of motion should be satisfied even if we put multiple boundaries in the flat spacetime. For example, we can construct a vacuum restricted to the region

$$a < \tilde{x}_1 - \tilde{x}_0 < b. \qquad (4.12)$$

This represents a spacetime where a finite interval moving at the speed of light as in the left figure of figure 6.

An explicit construction of this spacetime is to start with the space-like Liouville term (4.4) as well as the time-like Liouville term

$$\nu \int dz^2 \; e^{-2\beta X^0}, \qquad (4.13)$$

where $Q = \beta - 1/\beta$. After boost we obtain $T_{\text{closed}} \sim \mu e^{b\gamma(\tilde{x}_0-\tilde{x}_1)} + \nu e^{\beta\gamma(\tilde{x}_1-\tilde{x}_0)}$. Thus if we assume an appropriate limit of μ/ν we indeed find the restriction (4.12).

A more ambitious example may be the spacetime with two different types of the null boundaries i.e. $\tilde{x}^+ > 0$ and $\tilde{x}^- > 0$ at the same time (see the right figure of figure 6).

---

[25]Note also that the dilaton is constant after the boost. In the flat background with the null linear dilaton, only 16 supersymmetries are preserved [65].
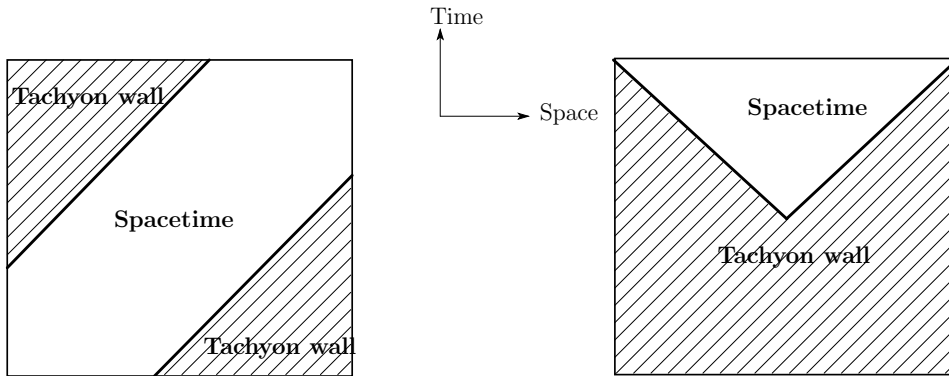
Figure 6: Examples of spacetimes with two null boundaries induced by closed string tachyon condensation. The two tachyon walls move in the same (or opposite) direction in the left (or right) spacetime.

Except the point $\tilde{x}^+ = \tilde{x}^- = 0$, the equation of motion is clearly satisfied as in the previous argument. However, since near the origin the boundaries coincide with each other, there is a possibility that we have to modify the solution to take stringy backreactions into account. If we neglect this issue, it is clear that this background describes a universe created at $t = x = 0$ and expands at the speed of light. This construction of the spacetime via the closed string tachyon condensation suggests the recently advertised idea that a spacetime is an emergent object. We leave further studies of this background for a future problem.

4.4 Effective action argument of null boundaries

It is also useful to see how the spacetime with the null boundary makes sense from the viewpoint of the effective low energy gravity theory with a closed string tachyon. Remember the usual caveat that we cannot trust this analysis quantitatively since the tachyon mass is of order string scale and that, strictly speaking, we have to take higher derivative terms into account.

We assume the following model as a candidate of an effective action for the $26D$ bosonic string

$$S = \frac{1}{2\kappa^2} \int (dx)^{26} \sqrt{-G} e^{-2\Phi} \left(R + 4(\partial_\mu \Phi)^2 - f(T)(\partial_\mu T)^2 - 2V(T) \right). \tag{4.14}$$

If we consider the particular case $f(T) = 1$, this is exactly the same as the one [69] considered by Yang and Zwiebach (see also [70, 71]). We extended this model to allow any function $f(T)$ with the requirement $f(0) = 1$. This is because physically we may be interested in the possibility $f(\infty) = 0$, which is motivated from the speculation that the complete tachyon condensation $T = \infty$ annihilates the spacetime and that there should be no degree of freedom as in the open string case [72].

The tachyon potential is supposed to satisfy the following properties [69]

$$V(0) = V'(0) = 0, \tag{4.15}$$

and

$$V(\infty) = V'(\infty) = 0. \tag{4.16}$$

The first conditions in (4.15) and (4.16) are required in order to satisfy the dilaton equation of motion. The vacuum $T = 0$ represents the ordinary (tachyonic) closed string vacuum, while another one $T = \infty$ represents the one after the tachyon condensation. The second conditions in (4.15) and (4.16) assure the absence of tad-pole of the tachyon field T. For example, the potential $V(T) = T^2 e^{-T}$ [70] proposed by Tseytlin indeed satisfies all of these conditions. However, it is fair to say that one of the assumptions $V(\infty) = 0$ has not been completely well-established (some counter evidences have been recently discussed in [73]). Even if $V(\infty) = 0$ is not correct, we believe that a certain (non-substantial) modification of our analysis below can be done to show that the null boundary is an allowed solution.

The equation of motions are given by

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - f(T)\partial_\mu T\partial_\nu T = 0,$$
$$2f(T)\nabla_\mu\partial^\mu T + f'(T)(\partial_\mu T)^2 - 4f(T)\partial_\mu\Phi\partial^\mu T - 2V'(T) = 0,$$
$$\nabla^2\Phi - 2(\partial_\mu\Phi)^2 - V(T) = 0. \tag{4.17}$$

Now we would like to confirm that the null boundary background indeed satisfies the equation of motions (4.17) . Suppose that the metric is flat $G_{\mu\nu} = \eta_{\mu\nu}$ and the dilaton and tachyon only depends on the light-cone coordinate x^+

$$\Phi = \Phi(x^+), \quad T = T(x^+). \tag{4.18}$$

Then the first equation in (4.17) leads to

$$2\partial_+^2\Phi = f(T)(\partial_+T)^2. \tag{4.19}$$

We can choose the tachyon field of the following form

$$T = \mu\, e^{\lambda x^+}, \tag{4.20}$$

which is the same as in the boosted Liouville model (4.6). The second and third equation in (4.17) are equivalent to

$$V(T) = V'(T) = 0. \tag{4.21}$$

Therefore if we take the limit $\lambda \to \infty$, then the tachyon profile (4.20) satisfies[26] (4.21) because of the properties (4.15) and (4.16).

We would also like to note that we can add the null dilaton $\Phi(x^+)_{\text{null}} = Qx^+$ for any Q with the equation of motions satisfied. In other word, we can start with the null dilaton background and cut off the strongly coupled region by putting the null boundary.

It can be easily checked that the above argument is also true for the model obtained by Tseytlin (setting $D = 26$) via the sigma-model approach [70]

$$S = \int dx^D \sqrt{g}\left(V(T)e^{\frac{4\Phi}{D-2}} + \alpha'F(T)(\partial_\mu T)^2 - \frac{\alpha'}{2}\left[R - \frac{4}{D-2}(\partial_\mu\Phi)^2\right]\right), \tag{4.22}$$

where V and F are explicitly given by

$$V(T) = -2T^2 e^{-2T}, \quad F(T) = 2(1 - T)e^{-2T}. \tag{4.23}$$

---

[26]In this limit the dilaton becomes trivial $\Phi(x^+) \to 0$ if we assume the profile $f(T) \sim e^{-T}$.

Finally we would like to comment on the total energy of the spacetime. As far as we assume (4.16) as well as (4.15), it is clear that the total energy is divergent as the kinetic energy is infinite due to the step function-like behavior of the tachyon field T. This is essentially because we boosted the tachyon wall infinitely and this will not be a serious problem as far as we start with this background from the beginning. The background cannot be created dynamically from the ordinary vacuum of flat spacetime. However, if the assumption $V(\infty) = 0$ in (4.16) is not correct and the tachyon vacuum has a negative energy, there is a possibility that the total energy is finite and even negative. Naively, our physical intuition influenced by the present knowledge of the open string tachyon condensation [72] suggests the second possibility i.e. $V(\infty) < 0$. However, this speculation may be wrong in the presence of dilaton as discussed in [69] and its final answer will be an important open problem.

4.5 D-brane analogue

As far as we consider an unstable D-brane (i.e. a D-brane with an open string tachyon) in type II or other string theories, we can apply the same boost argument in the presence of the null dilaton and the boundary Liouville term $\mu_B \int_{\partial\Sigma} dz \ e^{-bX_1}$ [74]. Then we can construct a D-brane with a null boundary in the same way.

5. Conclusions and discussions

More than half of this paper has been devoted to explore evidences for the conjectured scenario that unstable near horizon geometries of D-branes may decay into stable AdS bubbles with the same asymptotic geometry via the closed string tachyon condensation. In particular we examined the novel quantity called the entanglement entropy in both gravity and Yang-Mills side, which measures the degree of freedom. We show that the entropy decreases under the tachyon condensations in explicit examples as we expect. A new example discussed in this paper is the twisted AdS bubble obtained by the double Wick rotation of the rotating black 3-brane solution. This string theory background includes a closed string tachyon localized both in the IR region and in the north pole of the S^5. This tachyon is very similar to the one found in the twisted circle (or Melvin background). We can also say that the tachyon condensation process from the AdS twisted circle to the AdS twisted bubble is dual to the shift of the Wilson line expectation values from those for the bulk D3-branes to those for the fractional D3-branes if $k + M$ is even.

It is known that the AdS bubble has a lower energy than the one of the pure AdS_5 when we impose the anti-periodic boundary condition for fermions. The Casimir energy of the free $N = 4$ super Yang-Mills on $S^1 \times R^3$ with the same boundary condition for fermions agrees with the energy of AdS bubble in the gravity computation up to the factor of $\frac{4}{3}$. In the near extremal limit of the twisted AdS bubble example we find that this ratio is given by $\frac{9}{8}$ and thus becomes much more closer to 1.

We obtain a similar behavior also for the entanglement entropy. The free Yang-Mills/gravity ratio of the entropy becomes $\frac{2}{3}$ for the AdS bubble. Remarkably, in the near extremal limit the ratio becomes precisely 1. All of these results indicate that the

(twisted) AdS bubble is the true gravity dual geometry corresponding to the Yang-Mills theory on $S^1 \times R^3$ with the twisted boundary conditions.

In the final part of this paper we have discussed null spacetime boundaries in string theory. We observe that the tachyon walls or bubbles in a null linear dilaton background lead to such null boundaries after an infinite boost. They are exactly solvable time-dependent backgrounds since they are described by the Liouville theory before we take the boost. Because the metric of this spacetime is strictly flat except the sharp tachyon wall, it is natural to expect a direct string theoretic description of this spacetime without using the Liouville theory. In the light-cone gauge it may be described by the restriction $\tau > 0$ of the world-sheet time $\tau(= X^+)$. The covariant string description remains as a future problem.

Acknowledgments

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